

ON UPPER BOUNDS FOR THE AVAILABILITIES IN A FIXED TIME INTERVAL FOR MULTISTATE MONOTONE SYSTEMS

Bent Natvig, University of Oslo

Abstract

In Funnemark and Natvig (1985) upper and lower bounds for the availability and unavailability, to any level, in a fixed time interval are arrived at for multistate monotone systems based on corresponding information on the multistate components. The lower bounds are supposed to be good, the upper bounds very poor for long intervals. We will here show how the upper bounds can easily be improved inside the framework of the paper above.

1. Introduction and basic definitions

In a recent paper Funnemark and Natvig (1985) upper and lower bounds for the availability and unavailability, to any level, in a fixed time interval I are arrived at for multistate monotone systems based on corresponding information on the multistate components. These are assumed to be maintained and interdependent. Such bounds are of great interest when trying to predict the performance process of the system, noting that exact expressions are obtainable just for trivial systems. The bounds given in Funnemark and Natvig (1985) generalize the ones given in Natvig (1980) covering traditional binary theory.

The lower bounds given in these papers are supposed to be good, whereas most upper bounds are good only for short interval lengths. For long interval lengths most upper bounds may be very poor. It is the aim of this short note to indicate how each of these poor upper bounds can easily be improved, by just taking the infimum of all corresponding upper bounds calculated for each fixed point of time in the interval.

For easy reference we have to give a short introduction to some main concepts in multistate reliability theory. Let $S = \{0, 1, \dots, M\}$ be the set of states of the system ranging from the perfect functioning level M down to the complete failure level 0 . Let furthermore, $C = \{1, \dots, n\}$ be the set of components and S_i ($i=1, \dots, n$) the set of states of the i th component. We claim $\{0, M\} \subseteq S_i \subseteq S$. Let x_i ($i=1, \dots, n$) denote the performance level of the i th component and $\underline{x} = (x_1, \dots, x_n)$. It is assumed that the state, ϕ , of the system is given by the structure function $\phi = \phi(\underline{x})$.

Definition 1.1.

A system is a multistate monotone system (MMS) iff its structure ϕ satisfies

- i) $\phi(\underline{x})$ is nondecreasing in each argument
- ii) $\phi(\underline{0}) = 0$ and $\phi(\underline{M}) = M$ ($\underline{0} = (0, \dots, 0)$, $\underline{M} = (M, \dots, M)$).

Definition 1.2.

Let ϕ be the structure function of an MMS and let $j \in \{1, \dots, M\}$. A vector \underline{x} is said to be a minimal path (cut) vector to level j iff $\phi(\underline{x}) \geq j$ and $\phi(\underline{y}) < j$ for all $\underline{y} < \underline{x}$ ($\phi(\underline{x}) < j$ and $\phi(\underline{y}) \geq j$ for all $\underline{y} > \underline{x}$).

Definition 1.3.

The performance process of the i th component ($i=1, \dots, n$) is a stochastic process $\{X_i(t), t \in [0, \infty)\}$, where $X_i(t)$ is a r.v. which takes values in S_i . The performance process of an MMS with structure function ϕ is a stochastic process $\{\phi(\underline{X}(t)), t \in [0, \infty)\}$, where $\phi(\underline{X}(t))$ takes values in S .

Definition 1.4.

Let $j \in \{1, \dots, M\}$. The availability, $h_\phi^{j(I)}$, and the unavailability, $g_\phi^{j(I)}$, to level j in the time interval I for an MMS with structure function ϕ are given by

$$h_\phi^{j(I)} = P[\phi(\underline{X}(s)) > j \ \forall s \in I], \quad g_\phi^{j(I)} = P[\phi(\underline{X}(s)) < j \ \forall s \in I].$$

If $I = [t, t]$, we replace the I in the notation above by just t . Note that

$$h_\phi^{j(I)} + g_\phi^{j(I)} < 1, \text{ whereas}$$

$$h_\phi^{j(t)} + g_\phi^{j(t)} = 1 \quad \forall t \in I.$$

It is the former relation that causes the poor upper bounds for long intervals I .

2. Improved upper bounds for the availabilities and unavailabilities in a fixed time interval

The bounds for $h_\phi^{j(I)}$ and $g_\phi^{j(I)}$ in Theorem 3.1 of Funnemark and Natvig (1985) are supposed to be good even for fairly long intervals I , but seems of little practical value due to the complexity of the

bounds. As an illustration of the technique of improving the very poor upper bounds we present the improved version of Corollary 3.6 of the above mentioned paper:

Theorem 2.1

Let (C, ϕ) be an MMS with the marginal performance processes of its components being independent and each of them associated in I .

Furthermore for $j \in \{1, \dots, M\}$ let $Y_k^j = (Y_{1k}^j, \dots, Y_{nk}^j)$, $k=1, \dots, n^j$ ($z_k^j = (z_{1k}^j, \dots, z_{nk}^j)$, $k=1, \dots, m^j$) be its minimal path (cut) vectors to level j . Also denote the availability and unavailability to level j in I for the i th component by $p_i^{j(I)}$ and $q_i^{j(I)}$ respectively and introduce the $n \times M$ matrices

$$\underline{P}_\phi^{(I)} = \{p_i^{j(I)}\}_{\substack{i=1, \dots, n \\ j=1, \dots, M}} \quad \underline{Q}_\phi^{(I)} = \{q_i^{j(I)}\}_{\substack{i=1, \dots, n \\ j=1, \dots, M}}.$$

Define

$$\lambda_\phi^{j'}(\underline{P}_\phi^{(I)}) = \max_{1 \leq k \leq n^j} \prod_{i=1}^n p_i^{y_{ik}^j(I)} \quad \bar{\lambda}_\phi^{j'}(\underline{Q}_\phi^{(I)}) = \max_{1 \leq k \leq m^j} \prod_{i=1}^n q_i^{z_{ik}^{j+1}(I)}$$

$$\lambda_\phi^{j*}(\underline{P}_\phi^{(I)}) = \prod_{k=1}^{m^j} \prod_{i=1}^n p_i^{z_{ik}^{j+1}(I)} \quad \bar{\lambda}_\phi^{j*}(\underline{Q}_\phi^{(I)}) = \prod_{k=1}^{n^j} \prod_{i=1}^n q_i^{y_{ik}^j(I)}$$

$$B_\phi^j(\underline{P}_\phi^{(I)}) = \max_{j \leq k \leq M} \{ \max[\lambda_\phi^{j'}(\underline{P}_\phi^{(I)}), \lambda_\phi^{j*}(\underline{P}_\phi^{(I)})] \}$$

$$\bar{B}_\phi^j(\underline{Q}_\phi^{(I)}) = \max_{1 \leq k \leq j} \{ \max[\bar{\lambda}_\phi^{j'}(\underline{Q}_\phi^{(I)}), \bar{\lambda}_\phi^{j*}(\underline{Q}_\phi^{(I)})] \}$$

Then

$$B_\phi^j(\underline{P}_\phi^{(I)}) < h_\phi^{j(I)} < \inf_{t \in I} [1 - \bar{B}_\phi^j(\underline{Q}_\phi^{(t)})] < 1 - \bar{B}_\phi^j(\underline{Q}_\phi^{(I)})$$

$$\bar{B}_\phi^j(\underline{Q}_\phi^{(I)}) < g_\phi^{j(I)} < \inf_{t \in I} [1 - B_\phi^j(\underline{P}_\phi^{(t)})] < 1 - B_\phi^j(\underline{P}_\phi^{(I)})$$

Here $\prod_{i=1}^n a_i \stackrel{\text{def}}{=} 1 - \prod_{i=1}^n (1-a_i)$. The upper bounds to the right are the ones of Funnemark and Natvig (1985).

Proof

We give the proof for the improved upper bound of $h_{\phi}^{j(I)}$, the corresponding proof for $g_{\phi}^{j(I)}$ is completely similar. By applying Corollary 3.6 of Funnemark and Natvig (1985) for each fixed $t \in I$, we get

$$h_{\phi}^{j(I)} \leq \inf_{t \in I} h_{\phi}^{j(t)} \leq \inf_{t \in I} [1 - \bar{B}_{\phi}^j(Q_{\phi}^{(t)})].$$

Noting that for each fixed $t \in I$

$$\bar{B}_{\phi}^j(Q_{\phi}^{(t)}) \geq \bar{B}_{\phi}^j(Q_{\phi}^{(I)}),$$

the proof is completed.

It should finally be admitted that these improved upper bounds can be poor as is realized from the proof above. A case study, where these bounds enter, is given in Natvig et.al. (1986).

Acknowledgement

We are thankful to Renato Monteiro, a student of professor Richard Barlow, for a thought-provoking comment leading to this paper.

References.

Funnemark, E. and Natvig, B. (1985) Bounds for the availabilities in a fixed time interval for multistate monotone systems. Adv. Appl. Prob. 17, 638-655.

Natvig, B. (1980) Improved bounds for the availability and unavailability in a fixed time interval for systems of maintained, interdependent components. Adv. Appl. Prob. 12, 200-221.

Natvig, B., Sørmo, S. and Holen A.T. (1986) Multistate reliability theory - a case study. Adv. Appl. Prob. 18, to appear.